# ADVANCED COUNTING TECHNIQUES

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# ADVANCED COUNTING TECHNIQUES

- Many counting problems cannot be solved easily using the methods discussed in previous lecture about combinatorics. One such problem is: How many bit strings of length $n$ *do not contain two consecutive zero*? To solve this problem, let $a_n$ be the number of such strings of length $n$. An argument can be given that shows that the sequence $\{a_n\}$ satisfies the recurrence relation $a_{n+1} = a_n + a_{n-1}$ and the initial conditions $a_1 = 2$ and $a_2 = 3$.

# ADVANCED COUNTING TECHNIQUES

- This recurrence relation and the initial conditions determine the sequence $\{a_n\}$. *Moreover,* $a_n$ *explicit formula can be found for an from the equation* relating the terms of the sequence. As we will see, a similar technique can be used to solve many different types of counting problems.

# ADVANCED COUNTING TECHNIQUES

- We will discuss two ways that recurrence relations play important roles in the study of algorithms. First, we will introduce an important algorithmic paradigm known as dynamic programming. Algorithms that follow this paradigm break down a problem into overlapping sub problems. The solution to the problem is then found from the solutions to the sub problems through the use of a recurrence relation. Second, we will study another important algorithmic paradigm, divide-and-conquer.

# ADVANCED COUNTING TECHNIQUES

- Algorithms that follow this paradigm can be used to solve a problem by recursively breaking it into a fixed number of non overlapping sub problems until these problems can be solved directly. The complexity of such algorithms can be analyzed using a special type of recurrence relation. In this lecture we will discuss a variety of divide-and-conquer algorithms and analyze their complexity using recurrence relations.

# ADVANCED COUNTING TECHNIQUES

- We will also see that many counting problems can be solved using formal power series, called generating functions, where the coefficients of powers of $x$ *represent terms of the sequence* we are interested in. Besides solving counting problems, we will also be able to use generating functions to solve recurrence relations and to prove combinatorial identities.

# ADVANCED COUNTING TECHNIQUES

- Many other kinds of counting problems cannot be solved using the previous techniques, such as: How many ways are there to assign seven jobs to three employees so that each employee is assigned at least one job? How many primes are there less than 1000?

# ADVANCED COUNTING TECHNIQUES

- Both of these problems can be solved by counting the number of elements in the union of sets. We will develop a technique, called the principle of inclusion-exclusion, that counts the number of elements in a union of sets, and we will show how this principle can be used to solve counting problems.

# APPLICATIONS OF RECURRENCE RELATIONS

- A recursive definition of a sequence specifies one or more initial terms and a rule for determining subsequent terms from those that precede them. Also, a rule of the latter sort (whether or not it is part of a recursive definition) is called a recurrence relation and that a sequence is *called a solution of a recurrence relation if its terms satisfy the* recurrence relation.

# APPLICATIONS OF RECURRENCE RELATIONS

- In this lecture we will show that such relations can be used to study and to solve counting problems. For example, suppose that the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in $n$ hours? To solve this problem, let $a_n$ be the number of bacteria at the end of $n$ hours. Because the number of bacteria doubles every hour, the relationship $a_n = 2a_{n-1}$ holds whenever $n$ is a positive integer.

# APPLICATIONS OF RECURRENCE RELATIONS

- This recurrence relation, together with the initial condition $a_0 = 5$, uniquely determines $a_n$ for all nonnegative integers $n$. We can find a formula for $a_n$ using the iterative approach, namely that
  - $a_n = 5 \cdot 2^n$
- for all nonnegative integers $n$.

# MODELING WITH RECURRENCE RELATIONS

- We can use recurrence relations to model a wide variety of problems, such as finding compound interest, counting rabbits on an island, determining the number of moves in the Tower of Hanoi puzzle, and counting bit strings with certain properties.

# LINEAR RECURRENCE RELATIONS

**Definition 1**. A *linear homogeneous recurrence relation of degree* $k$ *with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k = 0$.

# LINEAR RECURRENCE RELATIONS

- The recurrence relation in the definition is linear because the right-hand side is a sum of previous terms of the sequence each multiplied by a function of $n$. The recurrence relation is homogeneous because no terms occur that are not multiples of the $a_j$'s. The coefficients of the terms of the sequence are all constants, rather than functions that depend on $n$. The degree is $k$ because an is expressed in terms of the previous $k$ terms of the sequence.

# LINEAR RECURRENCE RELATIONS

- A consequence of the second principle of mathematical induction is that a sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the $k$ initial conditions

$$a_0 = C_0, \qquad a_1 = C_1, \ldots, a_{k-1} = C_{k-1}$$

# SOLVING LINEAR RECURRENCE RELATIONS: EXAMPLE

- The recurrence relation $P_n = 11P_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five.

# SOLVING LINEAR RECURRENCE RELATIONS: EXAMPLE

- The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear. The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous. The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients.

- Linear homogeneous recurrence relations are studied for two reasons. First, they often occur in modeling of problems. Second, they can be systematically solved.

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where $r$ is a constant. Note that $a_n = r^n$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- When both sides of this equation are divided by $r^{n-k}$ and the right-hand side is subtracted from the left, we obtain the equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

- Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ $n$ is a solution if and only if $r$ is a solution of this last equation.

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- We call this the characteristic equation of the recurrence relation. The solutions of this equation are called the characteristic roots of the recurrence relation. As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- We will first develop results that deal with linear homogeneous recurrence relations with constant coefficients of degree two. Then corresponding general results when the degree may be greater than two will be stated. Because the proofs needed to establish the results in the general case are more complicated, they will not be given here.

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

We now turn our attention to linear homogeneous recurrence relations of degree two. First, consider the case when there are two distinct characteristic roots.

The case when there are two equal characteristic roots is left as exercise.

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- THEOREM 1. Let $c_1$ and $c_2$ be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has two distinct roots $r_1$ and $r_2$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = a_1 r_1^n + a_2 r_2^2$$

for $n = 0, 1, 2, \ldots,$ where $a_1$ and $a_2$ are constants.

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

*Proof*: We must do two things to prove the theorem. First, it must be shown that if $r_1$ and $r_2$ are the roots of the characteristic equation, and $\alpha_1$ and $\alpha_2$ are constants, then the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation. Second, it must be shown that if the sequence $\{a_n\}$ is a solution, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some constants $\alpha_1$ and $\alpha_2$.

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- *Proof (cont.):* Now we will show that if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, then the sequence $\{a_n\}$ is a solution of the recurrence relation. Because $r_1$ and $r_2$ are roots of $r_2 - c_1 r - c_2 = 0$, it follows that

$$r_1^2 = c_1 r_1 + c_2, \quad r_2^2 = c_1 r_2 + c_2.$$

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- *Proof (cont.):* From these equations, we see that

$$
\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \ \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \ \alpha_2 r_2^{n-2}) \\
&= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\
&= \alpha_1 r^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 = \alpha_1 r_1^n + \ \alpha_2 r_2^n = a_n
\end{aligned}
$$

This shows that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \ \alpha_2 r_2^n$ is a solution of the recurrence relation.

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- *Proof (cont.):* To show that every solution $\{a_n\}$ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots,$ for some constants $\alpha_1$ and $\alpha_2$, suppose that $\{a_n\}$ is a solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold. It will be shown that there are constants $\alpha_1$ and $\alpha_2$ such that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies these same initial conditions.

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- *Proof (cont.):* The requires that

$$a_0 = C_0 = \alpha_1 + \alpha_2$$
$$a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2$$

- We can solve these two equations for $\alpha_1$ and $\alpha_2$. From the first equation it follows that $\alpha_2 = C_0 - \alpha_1$. Inserting this expression into the second equation gives

$$C_1 = \alpha_1 r_1 + (C_0 - \alpha_1) r_2.$$

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- *Proof (cont.):* Hence, $C_1 = \alpha_1(r_1 - r_2) + C_0 r_2$.

This shows that

$$\alpha_1 = C_1 - C_0 r_2 r_1 - r_2 \text{ and}$$

$$\alpha_2 = C_0 - \alpha_1 = C_0 - C_1 - C_0 r_2 r_1 - r_2 = C_0 r_1 - C_1 r_1 - r_2,$$

Where these expressions for $\alpha_1$ and $\alpha_2$ depend on the first that $r_1 = r_2$. (When $r_1 = r_2$, this theorem is not true.) Hence, whith these values for $\alpha_1$ and $\alpha_2$, the sequence $\{a_n\}$ with $\alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the two initial conditions.

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- *Proof (cont.):* We know that $\{a_n\}$and $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ are both solutions of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and both satisfy the initial conditions when $n = 0$ and $n = 1$.Because there is a unique solution of a linear homogeneous recurrence relation of degree two with two initial conditions, it follows that the two solutions are the same, that is, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all non negative integers $n$.

# SOLVING LINEAR HOMOGENEOUS RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- *Proof (cont.):* We have completed the proof by showing that a solution of the linear homogeneous recurrence relation with constant coefficients of degree two must be of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ , where $\alpha_1$ and $\alpha_2$ are constants

# EXAMPLE

- What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

*Solution:* Theorem 1 can be used to solve this problem. The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$. Its roots are $r = 2$ and $r = -1$. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 2^n + \alpha_2(-1)^n$, for some constants $\alpha_1$ and $\alpha_2$.

# EXAMPLE

- *Solution:* From the initial conditions, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2,$$

$$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1).$$

Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$. Hence, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$$a_n = 3 \cdot 2^n - (-1)^n.$$

# EXAMPLE

- Find an explicit formula for the Fibonacci numbers.

*Solution*: Recall that the sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ and also satisfies the initial conditions $f_0 = 0$ and $f_1 = 1$. The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}.$$

- *Solution (cont.):* Therefore, from Theorem 1 itfollows that the Fibonacci numbers are given by
  - $f_n = \alpha_1 \left(\frac{(1+\sqrt{5})}{2}\right)^n + \alpha_2 \left(\frac{(1-\sqrt{5})}{2}\right)^n,$
- for some constants $\alpha_1$and $\alpha_2$. The initial conditions $f_0 = 0$ and $f_1 = 1$ can be used to find these constants.

- *Solution (cont.)*: We have $f_0 = \alpha_1 + \alpha_2 = 0$,

$$f_1 = \alpha_1 \left( \frac{(1 + \sqrt{5})}{2} \right)^n + \alpha_2 \left( \frac{(1 - \sqrt{5})}{2} \right)^n = 1.$$

- The solution to these simultaneous equations for $\alpha_1$ and $\alpha_2$ is

$$\alpha_1 = \frac{1}{\sqrt{5}}, \alpha_2 = -\frac{1}{\sqrt{5}}.$$

- Consequently, the Fibonacci numbers are given by

$$f_n = \frac{1}{\sqrt{5}} \left( \frac{(1+\sqrt{5})}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{(1-\sqrt{5})}{2} \right)^n.$$

# HOMEWORK: EXERCISES 2, 4, 6 ON P. 524;